# Partially ordered sets

Future Mathematicians Programme - Mathematical structures exploration
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**Idea.** *Partially ordered sets* capture the idea of order. A partially ordered set is a set equipped with a way to tell if an element is smaller than another one. Basic examples are given by numbers with the usual $\leq$ relation. But partially ordered sets can be way more general than this, managing to capture also general flowcharts. They naturally arise everywhere in mathematics and beyond.

**Definition.** A ***relation*** $\leq$ on a set $X$ is a subset of $X \times X$. This means that we have a way to say whether an element $x \in X$ is related to another element $y \in X$. Note that this is a qualitative property (rather than quantitative) of the two elements $x$ and $y$.

A ***preordered set*** is a set $X$ equipped with a relation $\leq$, denoted as $(X, \leq)$, such that all the following properties hold:

(1) *reflexivity:* for every $x \in X$, $x \leq x$;
(2) *transitivity:* for every $x, y, z \in X$, if $x \leq y$ and $y \leq z$ then $x \leq z$.

A ***partially ordered set*** is a preordered set that also satisfies the following condition:

(3) *antisymmetry:* for every $x, y \in X$, if $x \leq y$ and $y \leq x$ then $x = y$;

A partially ordered set is ***total*** if all elements are comparable with each other, i.e. for every $x, y \in X$ we have that $x \leq y$ or $y \leq x$.

**Problem 1.** Show that the following are all preordered sets:

(a) the set $\mathbb{R}$ of real numbers with the usual $\leq$ relation;
(b) the power set of the set $\mathbb{N}$ of natural numbers, i.e. the set of all subsets of $\mathbb{N}$, equipped with the inclusion relation $\subseteq$;
(c) the set $\mathbb{N}$ of natural numbers with the divisibility relation $|$: $n|m$ precisely when $n$ is a divisor of $m$, i.e. there exists $a \in \mathbb{N}$ such that $n \cdot a = m$;
(d) the set of logical propositions, i.e. statements that can be true or false, equipped with sequent relation $\vdash$: $P \vdash Q$ precisely when every time $P$ is true also $Q$ needs to be true.

Which of the preordered sets above is a partially ordered set? Which of these are total?

**Definition.** Let $(X, \leq)$ be a preordered set and $x, y \in X$. The ***meet*** of $x$ and $y$ in $X$ is an element $x \wedge y \in X$ such that $x \wedge y \leq x$ and $x \wedge y \leq y$, and for every other element $z \in X$ such that $z \leq x$ and $z \leq y$ we have that $z \leq x \wedge y$. The ***join*** of $x$ and $y$ in $X$ is an element $x \vee y \in X$ such that $x \leq x \vee y$ and $y \leq x \vee y$, and for every other element $z \in X$ such that $x \leq z$ and $y \leq z$ we have that $x \vee y \leq z$.

**Remark.** The meet of $x$ and $y$ in $X$ is the biggest (and thus the best) element that is smaller than both $x$ and $y$. The join of $x$ and $y$ in $X$ is the smallest (and thus the best) element that is bigger than both $x$ and $y$. The meet and the join are especially interesting when $x$ and $y$ are not comparable with each other. From the definition it follows that, in a partially ordered set, the meet and the join of $x$ and $y$ are unique (think about why!).

**Problem 2.** Describe the meets and the joins of two elements in all the preordered sets of Problem 1. You should find concepts you already know. These were apparently different concepts, belonging to different mathematical subjects, but they are actually all the same thing!

**Definition.** Let $(X, \leq)$ be a preordered set. The ***bottom*** of $X$ is (if it exists) an element 0 of $X$ such that for every $x \in X$ we have $0 \leq x$. The ***top*** of $X$ is (if it exists) an element 1 of $X$ such that for every $x \in X$ we have $x \leq 1$.

Let now $x \in X$. The ***complement*** of $x$ in $X$ is (if it exists) an element $x^* \in X$ such that $x \wedge x^* = 0$ and $x \vee x^* = 1$.

**Problem 3.** Describe the complements in the preordered sets (b) and (d) of Problem 1.

**Interesting links to other mathematical structures.**

A preordered set is precisely a ***category*** in which there is at most one morphism from one object to another, representing the $\leq$ relation.

Particular partially ordered sets equipped with meets and joins, called frames, generalize ***topological spaces***.